Infinite multiplicity for inhomogeneous supercritical problem in entire space *

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Abstract

In this paper, we will prove the existence of infinitely many positive solutions to the following supercritical problem by using the Liapunov-Schmidt reduction method and asymptotic analysis:

$$\begin{cases} \Delta u + u^p + f(x) = 0, \quad u > 0 \;\text{ in } R^n, \\ \lim_{|x|\to\infty} u(x) \to 0. \end{cases}$$

Keywords: Critical exponents; Linearized operators; Supercritical problem.

1. Introduction and statement of the main results

The purpose of this paper is to establish the existence of infinitely many positive solutions to the following inhomogeneous problem

$$\begin{cases} \Delta u + u^p + f(x) = 0, \\ u > 0 \;\text{ in } R^n, \; \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.1}$$

Where $n \geq 3, p > \frac{n+2}{n-2}, \Delta = \sum_{i=1}^{n}(\partial^2/\partial x_i^2)$ is the Laplacian operator and $f(x) \in C^{0,\alpha}_{loc}(R^n)$ with $f \geq 0$ everywhere in R^n, $f \not\equiv 0$.

Inhomogeneous second-order elliptic equations defined in entire space arise naturally in probability theory in the study of stochastic processes. the eqns. (1.1) in particular appeared recently in a paper by Tzong-Yow Lee [9] establishing limit theorems for super-brownian motion. In that paper, existence results for the eqns. (1.1) were obtained in the case where the inhomogeneous terms is compactly supported and is dominated by a function of the form $\frac{C}{(1+|x|)^{(n-2)p}}$ where $C(n,p) > 0$ is sufficiently small. In addition to Lee [9], (1.1) has been studied by Pokhozhaev [11] and Egnell and Kay [10]. Pokhozhaev obtained radial solutions when the inhomogeneous term is radially symmetric about the origin and satisfies certain integrability conditions. Also, Egenll and Kay worked on an equation similar to that of (1.1) but have a small positive

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parameter ε as a coefficient to inhomogeneous term $f(x)$. In a recent paper [1], existence results were obtained for a large class of functions for the inhomogeneous term, i.e.,

$$0 \leq f(x) \leq \frac{p-1}{[p(1+|x|^2]^{\frac{p}{p-1}}} L^p$$

for all $x \in R^n$, where

$$L = \Big[\frac{2}{p-1}(n-2-\frac{2}{p-1})\Big]^{\frac{1}{p-1}}.$$

Using sub-super solution method, Bae and Ni established the following infinite multiplicity result for the equation

$$\begin{cases} \Delta u + u^p + \mu f(x) = 0, \quad u > 0 \ \text{ in } R^n, \\ \lim_{|x|\to\infty} u(x) \to 0. \end{cases}$$

where $\mu > 0$ is a parameter.

Theorem A[1]**:** (i) Let $p > p_c$. Suppose that near ∞

$$\max(\pm f(x), 0) \leq |x|^{-q_\pm},$$

where $q_+ > n - \lambda_2$ and $q_- > n - \lambda_2 - \frac{2}{p-1}$. Then, there exists $\mu_* > 0$ such that for every $\mu \in (0, \mu_*)$, equation (1.2) possesses infinitely many solutions with asymptotic behavior $L|x|^{-m}$ at ∞

(ii) Let $p = p_c$. Then, the conclusion in (i) holds if we assume in addition that either f has a compact support in R^n or f does not change sign in R^n, where

$$p_c = \begin{cases} \frac{(n-2)^2 - 4n + 4\sqrt{n^2-(n-2)^2}}{(n-2)(n-10)} & n > 10, \\ \infty & 3 \leq n \leq 10. \end{cases}$$

and

$$\lambda_2 = \frac{(n-2-\frac{4}{p-1}) + \sqrt{(n-2-\frac{4}{p-1})^2 - 8(n-2-\frac{2}{p-1})}}{2}$$

Besides, for a more general case (i.e., $u^p + \mu f$ is replaced by $K(x)u^p + \mu f$), similar results are obtained, see [2, 8, 12]. However, these results are established require $p > p_c, n \geq 11$. But the case of $n \leq 10$ and $\frac{n+2}{n-2} < p < p_c$ is **open**. Note that in this case, the method of sub-super solutions breaks down. In this paper, under reasonable conditions on f, we establish that when $p > \frac{n+1}{n-3}$, (1.1) has a continuum of solutions.

The main difficulties in establishing existence of (1.1), in addition to the noncompactness of the domain and the presence of an inhomogeneous term, are the lack of (local) sobolev embedding suitably fit to a weak formulation of this problem. So direct tools of the calculus of variation are not appropriate for (1.1).

Instead of using sub-super solution method (which limits the applicability on the exponent p), we use asymptotic analysis and Liapunov-Schmidt reduction method to prove Theorem 1. This is based on the construction of a sufficiently good approximation solution. It is well known that the problem

$$\Delta w + w^p = 0 \ \text{ in } R^n \tag{1.2}$$

where $p \geq \frac{n+2}{n-2}$, possesses a positive radially symmetric solution $w(|x|)$ which reduces to the equation

$$w'' + \frac{n-1}{r}w' + w^p = 0. \tag{1.3}$$

This equation can be analyzed through phase plane analysis after a transformation introduced by Fowler [9] in 1931: $v(s) = r^{\frac{2}{p-1}}w(r), r = e^s$, which transforms equation (1.3) into the autonomous ODE

$$v'' + \alpha v' - \beta v + v^p = 0 \tag{1.4}$$

where

$$\alpha = n - 2 - \frac{4}{p-1}, \beta = \frac{2}{p-1}(n - 2 - \frac{2}{p-1}).$$

Since α, β are positive for $p > \frac{n+2}{n-2}$, the Hamiltonian energy

$$E(v) = \frac{1}{2}v'^2 + \frac{1}{p+1}v^{p+1} - \frac{\beta}{2}v^2$$

strictly decrease along trajectories. Using this it is easy to see the existence of a heteroclinic orbit which connects the equilibria (0,0) and $(0, \beta^{\frac{1}{p-1}})$ in the phase plane (v, v'), which corresponding respectively to a saddle point and an attractor. A solution of (1.4) corresponding this orbit satisfies $v(-\infty) = 0, v(+\infty) = \beta^{\frac{1}{p-1}}$ and $w(r) = r^{-\frac{2}{p-1}}v(\log r)$ solves (1.3) and is bounded at $r = 0$. Then all radial solutions of (1.3) defined in all R^n have the form

$$w_\lambda(x) := \lambda^{\frac{2}{p-1}}w(\lambda|x|), \quad \lambda > 0.$$

We denote in what follows by $w(x)$ the unique positive radial solution

$$\Delta w + w^p = 0 \ \text{ in } \ R^n, \quad w(0) = 1.$$

At main order one has

$$w(r) \sim Lr^{-\frac{2}{p-1}} \ \text{ as } r \to \infty,$$

which implies that this behavior is actually common to all solutions $w_\lambda(x)$.

By the change of variable $\lambda^{-\frac{2}{p-1}}u(\frac{x}{\lambda})$ the (1.1) becomes

$$\Delta u + u^p + f_\lambda(x) = 0, \ \text{ in } \ R^n \ \ u > 0, \quad \lim_{|x| \to +\infty} u(x) = 0, \tag{1.5}$$

where $f_\lambda(x) = \lambda^{-\frac{2p}{p-1}}f(\frac{x}{\lambda})$. In fact, if f is assumed to satisfy the asymptotic behavior

$$f(x) = o(|x|^{-\frac{2p}{p-1}}) \ \text{ as } \ |x| \to +\infty,$$

then we observe that away from the origin $f_\lambda(x) \to 0$ as $\lambda \to 0$. Thus (1.1) may be regarded, away from the origin, as small perturbations of problem (1.2) when $\lambda > 0$ is sufficiently small. From (1.5), we find that (1.1) "hide" a parameter which indexes a continuum of solutions which asymptotically vanish over compact sets.

Our main result is as follows:

Theorem 1. Let $p > \frac{n+1}{n-3}, n \geq 4$. Assume that

$$f(x) = \eta(x) f_1(x),$$

where $\eta(x) \in C_0^\infty(R^n), 0 \leq \eta(x) \leq 1$,

$$\eta(x) = 0 \ \text{ for } \ |x| \leq R_1, \eta(x) = 1 \ \text{ for } |x| \geq R_1 + 1,$$

($R_1 > 0$ fixed large enough) and $f_1(x) \in C^{0,\alpha}(R^n), f_1(x) < |x|^{-\mu}$ with $\mu > 2 + \frac{2}{p-1}$. Then problem (1.1) has a continuum of solutions $u_\lambda(x)$ (parameterized by $\lambda \leq \lambda_0$, where λ_0 is a fixed number) such that

$$\lim_{\lambda \to 0} u_\lambda(x) = 0$$

uniformly in $R^n \setminus \{0\}$. The same results holds when $\frac{n+2}{n-2} < p \leq \frac{n+1}{n-3}$ provided that f is symmetric with respect to n coordinate axis, namely

$$f(x_1, ..., x_i, ..., x_n) = f(x_1, ... - x_i, ...x_n), \quad \text{for all } i = 1, ..., n.$$

The idea is to consider $w(x)$ as an approximation for a solution of (1.1), provided that $\lambda > 0$ is chosen small enough. To this end, we need to study the solvability of the operator $\Delta + pw^{p-1}$ in suitable weighted Sobolev space. Recently, this issue has been studied in Davila-del Pino-Musso [4] and Davila-Pino-Musso-Wei [5].

Throughout the paper, the symbol C denotes always a positive constant independent of λ, which could be changed from one line another. Denote $A \sim B$ if and only if there exist two positive numbers a, b such that $aA \leq B \leq bA$.

2. The solvability of linearized operator $\Delta + pw^{p-1}$

Our main concern in this section is to state the results concerning the existence of solution in certain weighted spaces for

$$\Delta\phi + pw^{p-1}\phi = h \ \text{ in } R^n, \tag{2.1}$$

where w is the radial solution to (1.2) and h is a known function having a specific decay at infinity. We are looking for a solution to (2.1) that is turn out to be a perturbation of w, it is rather natural to require that it has a decay at most the same as that of w, namely $O(|x|^{-\frac{2}{p-1}})$ as $|x| \to \infty$. Of course we would also like ϕ be bounded on compact sets. As a result, we shall assume that h behaves like this but with two powers subtracted, that is, $h = O(|x|^{-\frac{2}{p-1}-2})$ at infinity.

Now, define some weighted L^∞ norms as follows (adopted from [4,5]):

$$\|\phi\|_* = \sup_{|x|\leq 1} |x|^\sigma |\phi(x)| + \sup_{|x|\geq 1} |x|^{\frac{2}{p-1}} |\phi(x)|,$$

and

$$\|h(x)\|_{**} = \sup_{|x|\leq 1} |x|^{2+\sigma} |h(x)| + \sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}} |h(x)|,$$

where $\sigma > 0$ will be fixed later as needed.

The following lemmas and remarks on the solvability are due to Davila-del Pina-Musso [4] and Davila-del Pino-Musso-Wei [5]:

Lemma 2.1. Assume that $p > \frac{n+1}{n-3}, n \geq 4$. For $0 < \sigma < n-2$ there exists a constant $C > 0$ such that for any h with $\|h(x)\|_{**} < \infty$, equation (2.1) has a solution $\phi = T(h)$ such that T defines a linear map and

$$\|T(h)\|_* \leq C\|h\|_{**}.$$

For the sake of completeness, we give the main idea of the proof of Lemma 2.1 as follows (for the details, see [4, 5]):

Let $\Theta_k, k \geq 0$ be the eigenfunction of the Laplace-Beltrami operator $-\Delta_{s^{n-1}}$ on the sphere S^{n-1} with eigenvalues λ_k repeated according to their multiplicity, normalized so that they constitute an orthonormal system in $L^2(S^{n-1})$. We let Θ_0 be a positive constant, associated to the eigenvalue 0 and $\Theta_i, 1 \leq i \leq n$ is an appropriate multiple of $\frac{x_i}{|x|}$ which has eigenvalue $\lambda_i = n-1, 1 \leq i \leq n$. we repeat eigenvalues according to their multiplicity and we arrange them in an non-decreasing sequence. We recall that the set of eigenvalues is given by $\{j(n-2+j)|j \geq 0\}$. We write h as

$$h(x) = \Sigma_{k=0}^{k=+\infty} h_k \Theta_k(\theta) \tag{2.2}$$

and look for a solution ϕ to (2.1) in the form

$$\phi(x) = \Sigma_{k=0}^{k=+\infty} \phi_k \Theta_k(\theta). \tag{2.3}$$

Then

$$\phi_k'' + \frac{n-1}{r}\phi' + (pw^{p-1} - \frac{\lambda_k}{r^2})\phi = h_k \tag{2.4}$$

Equation (2.4) can be solved for each k separately:

a: If $k = 0$ and $p > \frac{n+2}{n-2}, \|h_0\|_{**} < +\infty$ then (2.4) has a solution ϕ_0 which depends linearly on h_0 and satisfies

$$\|\phi_0\|_* \leq C\|h_0\|_{**}. \tag{2.5}$$

Indeed in this case this solution is defined using the variation of parameters formula

$$\phi_0(r) := z_{1,0}(r)\int_1^r z_{2,0}h_0 s^{n-1}ds - z_{2,0}(r)\int_0^r z_{1,0}h_0 s^{n-1}ds, \tag{2.6}$$

where $z_{1,0}, z_{2,0}$ are two special linearly independent solution to (2.4) with $k = 0$ and $h_0 = 0$. More precisely, we take $z_{1,0} = rw' + \frac{2}{p-1}w$ and $z_{2,0}$ a linearly independent solution. Linearization shows that

$$z_{j,0}(r) = O(r^{-\frac{n-2}{2}}) \text{ as } r \to +\infty, \quad j = 1, 2,$$

while

$$z_{2,0} \sim r^{2-n} \text{ near } r = 0.$$

Using this and (2.6), we can easily get estimate (2.5)

b: If $k = 1, n \geq 4$ and $p > \frac{n+1}{n-3}, \|h_1\|_{**} < +\infty$, then we have

$$\|\phi_1\|_* \leq C\|h_1\|_{**}. \tag{2.7}$$

In this case, we have that the positive function $z_1 := -w'(r) > 0$ in $(0, +\infty)$ solves (2.4) with $k = 1$ and $h_1 = 0$. Using this, we then define $\phi_1(r)$ as

$$\phi_1(r) = -z_1(r) \int_1^r z_1(s)^{-2} s^{1-n} ds \int_0^s z_1(\tau) h_1(\tau) \tau^{n-1}. \tag{2.8}$$

Using this formula and by a simple computer, estimate (2.7) is easily obtained.

c: Let $k \geq 2$ and $p > \frac{n+2}{n-2}$. If $\|h_k\|_{**} < +\infty$ (2.4) has a unique solution ϕ_k with $\|\phi_k\|_* < +\infty$ and there exists $C_k > 0$ such that

$$\|\phi_k\|_* \leq C \|h_k\|_{**} \tag{2.9}$$

this case is simpler because the operator

$$L_k \phi = \phi'' + \frac{n-1}{r} \phi' + (p w^{p-1} - \frac{\lambda_k}{r^2}) \phi$$

satisfies the maximum principle in any interval of the form $(\delta, \frac{1}{\delta}), \delta > 0$. Indeed let $z = -w'$, so that $z > 0$ in $(0, +\infty)$ and it is a supersolution, because

$$L_k z = \frac{n - 1 - \lambda_k}{r^2} z < 0 \ \text{ in } \ (0, +\infty),$$

since $\lambda_k \geq 2n$ for $k \geq n$. We construct a supersolution ψ of the form

$$\psi = C_1 z + v, \quad v(r) = \frac{1}{r^\sigma + r^{\frac{2}{p-1}}}.$$

Choosing C_1 sufficiently large, we can check that

$$L_k \psi \leq -c \min(r^{-\sigma-2}, r^{-\frac{2}{p-1}-2}) \ \text{ in } (0, +\infty).$$

Using this, we can easily obtain (2.9).

The previous construction and (2.5), (2.7) and (2.9) imply that given an integer $m > 0$, if $\|h\| < +\infty$ satisfies $h_k \equiv 0 \ \ \forall k \geq m$ then there exists a solution ϕ to (3.1) that depends linearly with respect to h and moreover

$$\|\phi\|_* \leq C_m \|h\|_{**}$$

where C_m may depend on only m. Then by using a blow up argument, we can show that C_m can be chosen independently of m. For a detailed proof, we refer the interested readers to [5,6]. □

Remark 2.1. From the above proof, we know the operator T in Lemma 2.1 are constructed "by hand" by decomposing h and ϕ into suns of spherical harmonics where the coefficients are radial functions. The nice property is of course that w is radial, the problem decouples into an infinite collection of ODEs.

Remark 2.2. If $p \geq \frac{n+2}{n-2}$, linearized operator $\Delta + p w^{p-1}$ has a kernel, i.e., span$\{\frac{\partial w}{\partial x_1}, i = 1, 2, ...n\}$, in general Sobolev space. However, under suitable weighted Sobolev space, the linearized operator $\Delta + p w^{p-1}$ is invertible, i.e., the kernel is 0.

3. The proof of Theorem 1

Let $p > \frac{n+1}{n-3}$, we will prove Theorem 1 in this section. The main idea is to use Proposition 2.1 and a contraction mapping principle.

We look for a solution of (1.5) of the form $u = w + \phi$, which yields the following equation for ϕ

$$\Delta\phi + pw^{p-1}\phi = N(\phi) - f_\lambda(x),$$

where

$$N(\phi) = -(w+\phi)^p + w^p + pw^{p-1}\phi. \tag{3.1}$$

Using the operator T defined in Proposition 2.1, we are led to solving the fixed point problem

$$\phi = T(N(\phi) - f_\lambda(x)). \tag{3.2}$$

Firstly let us estimate $\|N(\phi)\|_{**,\lambda}$ depending on whether $p \geq 2$ or $p < 2$.

Case $p \geq 2$. In this case, we observe that

$$|N(\phi)| \leq C(w^{p-2}\phi^2 + |\phi|^p).$$

Let us work with $0 < \sigma \leq \frac{2}{p-1}$. Since

$$|\phi(x)| \leq C|x|^{-\frac{2}{p-1}}\|\phi\|_*, \quad \text{for all } |x| \geq 1,$$

and

$$w(x) \leq C(1+|x|)^{-\frac{2}{p-1}}, \text{ for all } x \in R^n,$$

so we have on one hand

$$\sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}} w^{p-2}|\phi|^2 \leq C\|\phi\|^2. \tag{3.3}$$

On the other hand,

$$|\phi| \leq C|x|^{-\sigma}\|\phi\|_*, \text{ for all } |x| \leq 1,$$

and therefore, We obtain

$$\sup_{|x|\leq 1} |x|^{2+\sigma} w(x)^{p-2}|\phi|^2 \leq \|\phi\|_*^2 \sup_{|x|\leq 1} |x|^{2-\sigma} = C\|\phi\|_*^2. \tag{3.4}$$

From (3.3) and (3.4) it follows that

$$\|w^{p-2}\phi^2\|_{**} \leq C\|\phi\|_*^2. \tag{3.5}$$

To estimate$\||\phi|^p\|_{**}$ we compute

$$\sup_{|x|\leq 1} |x|^{2+\sigma}|\phi(x)|^p \leq C\|\phi\|_*^p. \tag{3.6}$$

Similarly

$$\sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}}|\phi|^p \leq \|\phi\|_*^p. \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$\||\phi|^p\|_{**} \leq C\|\phi\|_*^p. \tag{3.8}$$

By (3.5) and (3.8) we have

$$\|N(\phi)\|_{**} \leq C(\|\phi\|_*^2 + \|\phi\|_*^p). \tag{3.9}$$

Case $p < 2$. In this case $|N(\phi)| \leq C|\phi|^p$ and hence, if $0 < \sigma \leq \frac{2}{p-1}$,

$$\sup_{|x|\leq 1} |x|^{2+\sigma}|\phi(x)|^p \leq C\|\phi\|_*^p \tag{3.10}$$

Similarly

$$\sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}}|\phi|^p \leq \|\phi\|_*^p. \tag{3.11}$$

From (3.10) and (3.11) it follows that for any $1 < p < 2$ and $0 < \sigma \leq \frac{2}{p-1}$,

$$\|N(\phi)\|_{**} \leq C\|\phi\|_*^p. \tag{3.12}$$

From (3.9) and (3.12) we have

$$\|N(\phi)\|_{**} \leq C(\|\phi\|_*^2 + \|\phi\|_*^p). \tag{3.13}$$

Now, we estimate $\|f_\lambda(x)\|_{**}$ as follows:

$$\sup_{|x|\leq 1} |x|^{2+\sigma}|f_\lambda(x)| = \sup_{\lambda R_1 < |x| \leq 1} |x|^{2+\sigma}|f_\lambda(x)| \leq R_1^{\mu-2-\sigma} \to 0 \quad \text{as} \quad R_1 \to +\infty, \tag{3.14}$$

provided that $\sigma = \frac{2}{p-1}$.

Similarly,

$$\sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}}|f_\lambda(x)| = o(\lambda). \tag{3.15}$$

So we have, as $\lambda \to 0$,

$$\|f_\lambda(x)\|_{**} = \sup_{|x|\leq 1} |x|^{2+\sigma}|f_\lambda(x)| + \sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}}|f_\lambda(x)| \leq R_1^{\mu-2-\sigma}. \tag{3.16}$$

We have already observed that $u = w + \phi$ is a solution of (1.5) if ϕ satisfies equation (3.2). Consider the set

$$F = \{\phi \in L^\infty / \ \|\phi\|_* \leq \rho\}$$

where $\rho \in (0, 1)$ is to be chosen (suitably small) and the operator

$$\hbar(\phi) = T(N(\phi) - f_\lambda(x)).$$

We now prove that $\hbar$ has a fixed point in F. For $\phi \in F$ we have

$$\begin{aligned} \|\hbar(\phi)\|_* &\leq C\|N(\phi)\|_{**} + C\|f\|_{**} \\ &\leq C(\|\phi\|_*^2 + \lambda^{-2}\|\phi\|_*^p + R_1^{\mu-2-\sigma}) \end{aligned}$$

by (3.9) and (3.10), if $\sigma = \frac{2}{p-1}$. Then we have

$$\|\hbar(\phi)\|_{*,\lambda} \leq C(\rho^2 + \rho^p + R_1^{\mu-2-\sigma}) \leq \rho,$$

if we choose ρ is small enough and R_1 is large enough. Hence $\hbar(F) \subset F$.

Now we show that $\hbar$ is a contraction mapping in F. Let us take ϕ_1, ϕ_2 in F, then

$$\|\hbar(\phi_1) - \hbar(\phi_2)\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_{**}. \tag{3.17}$$

Write

$$N(\phi_1) - N(\phi_2) = D_\phi N(\bar{\phi})(\phi_1 - \phi_2),$$

where $\bar{\phi}$ lies in the segment joining ϕ_1 and ϕ_2.

For $|x| \leq 1$,

$$|x|^{2+\sigma}|N(\phi_1) - N(\phi_2)| \leq |x|^2|D_\phi N(\bar{\phi})|\|\phi_1 - \phi_2\|_*,$$

while, for $|x| \geq 1$,

$$|x|^{2+\frac{2}{p-1}}|N(\phi_1) - N(\phi_2)| \leq |x|^2|D_\phi N(\bar{\phi})|\|\phi_1 - \phi_2\|_*.$$

Then we have

$$\|N(\phi_1) - N(\phi_2)\|_{**} \leq C\sup_x |x|^2|D_\phi N(\bar{\phi})|\|\phi_1 - \phi_2\|_*. \tag{3.18}$$

Directly from the definition of N, we compute

$$D_\phi N(\bar{\phi}) = p[(w + \bar{\phi})^{p-1} - w^{p-1}]. \tag{3.19}$$

Thus

$$|D_\phi N(\bar{\phi})| \leq C(w^{p-2}|\bar{\phi}| + |\bar{\phi}|^{p-1}).$$

For all x we have

$$|x|^2 w^{p-2}|\bar{\phi}| \leq C(\|\phi_1\|_* + \|\phi_2\|_*) \leq C\rho. \tag{3.20}$$

Similarly, for all x

$$|x|^2|\bar{\phi}|^{p-1} \leq C(\|\phi_1\|_*^{p-1} + \|\phi_2\|_*^{p-1}) \leq C\rho^{p-1}. \tag{3.21}$$

Estimates (3.13)-(3.15) show that

$$\sup_x |x|^2|D_\phi N(\bar{\phi})| \leq C(\rho + \rho^{p-1}). \tag{3.22}$$

Gathering relations (3.11), (3.12), (3.16) we conclude that $\hbar$ is a contraction mapping in F, and hence a fixed point in this region indeed exists. So $w + \phi_\lambda$ is solution of

$$\Delta u + u^p + f_\lambda(x) = 0, \quad \text{in} \quad R^n \quad u > 0, \quad \lim_{|x|\to+\infty} u(x) = 0, \tag{3.23}$$

and

$$\phi_\lambda(x) \leq C \quad \text{for all} \quad x \in R^n \setminus \{0\}.$$

Thus $u_\lambda(x) = \lambda^{\frac{2}{p-1}}(w(\lambda x) + \phi_\lambda(\lambda x))$ is a continuum of (1.1) and

$$\lim_{\lambda\to 0} u_\lambda(x) = 0$$

uniformly in $R^n \setminus \{0\}$. This finishes the proof of the theorem 1. □

To conclude, in this paper, instead of using sub-super solution method (which limits the applicability on the exponent p), we use asymptotic analysis and Liapunov-Schmidt reduction method to solve a open problem.

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